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Exact Lyapunov exponent for infinite products of random matrices

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Abstract. Despite significant work since the original paper by H Furstenberg in the early 60s, explicit formulae for Lyapunov exponents of infinite products of random matrices are available only in a very few cases.

In this work, we give a rigorous explicit formula for the Lyapunov exponent for some binary infinite products of random 2×2 real matrices. All these products are constructed using only two types of matrices, A and B, which are chosen according to a stochastic process. The matrix A is singular, namely its determinant is zero. This formula is derived by using a particular decomposition for the matrix B, which allows us to write the Lyapunov exponent as a sum of convergent series. The key point is the computation of all the integer powers of B, which is achieved by a suitable change of frame. The computation then follows by looking at each of the special types of B (hyperbolic, parabolic and elliptic). Finally, we show, with an example, that the Lyapunov exponent is a discontinuous function of the given parameter.

1. Introduction

The product of random matrices appears in the study of disordered systems [1] as well as in the context of dynamical systems [2]. The Lyapunov exponents are one of the tools to study these products. They are related to physical quantities in disordered systems [3]. For example, in the tight-binding model or Anderson model [3], the localization length of the wavefunction is proportional to the inverse of the Lyapunov exponent. In dynamical systems theory, products of random matrices often arise as a non-trivial approximation which strongly mimics chaotic behaviour in deterministic systems.

In spite of important and numerous results obtained in the theory of random matrices [4–6], there is no general method for calculating the Lyapunov exponents [7–9], although for a few examples there is an explicit formula [10, 11].

In this paper, we present some examples of products of random matrices, where we were able to determine the Lyapunov exponents as a sum of explicitly convergent series. In all these examples, we deal with infinite binary random products, built with only two types of 2×2 real matrices, A and B, which are chosen according to a stochastic Bernoulli or Markovian process and one of the matrices, say A, is singular, i.e. its determinant is zero. Markovian processes are chosen to mimic the correlations existing in dynamical systems exhibiting weak chaos [12]. The key to our study is the use of a particular decomposition when the matrix B is non-singular (but if B is singular, the calculation can be done directly). The Bernoulli case was studied by Pincus [13], who gave asymptotic results

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for the Lyapunov exponent, see (2.6) which, however, do not enable an actual computation; see remark 2.3 below (also Derrida and Hilhorst [14] have obtained a similar formula for a particular product). Instead, we give an explicit form of the terms of that series. The key point is the computation, for all powers B^n ($n > 1$), of the matrix B of the corresponding entries $b_{ij}(n)$. This is achieved using a normal form given by (4.1) in each case of interest, (4.10) and (4.11). In some cases, this permits either an explicit summation of the series, or a control of the convergence, therefore leading to approximate results. Furthermore, we could show that the convergence of the series is exponentially fast.

The Markovian products are treated using (2.7), a formula that generalizes the corresponding formula of Pincus, (2.6).

This paper is organized as follows. In section 2, we derive the general formula for the largest Lyapunov exponent γ. Section 3 is devoted to the simple case where B is a singular matrix. In section 4 we perform the decomposition of B which, in turn, leads to the decompostion of B^n and in section 5 we derive the expression of γ. In the last section, we analyse the continuity of γ as a function of parameters.

2. Lyapunov exponent, general formula

We consider A and B, 2×2 real matrices, where A is a singular matrix. Now let $\mathcal{P}_N$,

$$\mathcal{P}_N = X_N X_{N-1} \ldots X_2 X_1$$

be a binary product of N matrices, where the matrix X_i is either A or B, the choice being made by a stochastic process. In the Bernoulli case, we have $X_i = A$ with probability p ($0 < p < 1$) and $X_i = B$ with probability $q = 1 - p$ ($i \geqslant 1$). In the Markovian case, the transition probabilities are given by

$$\begin{aligned} &\Pr(X_{n+1} = A/X_n = B) = p_1 \qquad \Pr(X_{n+1} = B/X_n = A) = p_2 \\ &\Pr(X_{n+1} = B/X_n = B) = 1 - p_1 = q_1 \qquad \Pr(X_{n+1} = A/X_n = A) = 1 - p_2 = q_2 \end{aligned} \tag{2.1}$$

and

$$\Pr(X_1 = A) = \frac{p_1}{p_1 + p_2} = p_0 \qquad \Pr(X_1 = B) = \frac{p_2}{p_1 + p_2} = q_0 \tag{2.2}$$

with $0 < p_1 < 1$ and $0 < p_2 < 1$.

By definition, the Lyapunov exponent γ is

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \log \| \mathcal{P}_N \| . \tag{2.3}$$

γ is independent of the choice of the norm $\| \cdot \|$.

Since the matrix A is singular, it can be written by a change of basis in one of the following forms:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \tag{2.4}$$

or

$$A = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} . \tag{2.5}$$

If A is of the type (2.5), the Lyapunov exponent is $\gamma = -\infty$. This is straightforward to show, since $A^2 = 0$.

Therefore we suppose, without loss of generality, that A has the form (2.4), i.e. $A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$.

If we write B^n in the form $B^n = \begin{pmatrix} b_{11}(n) & b_{12}(n) \\ b_{21}(n) & b_{22}(n) \end{pmatrix}$, then the result obtained by Pincus [13], in the Bernoulli case, follows:

$$\gamma = p \log |\lambda| + \sum_{n=1}^{\infty} p^2(1-p)^n \log |b_{11}(n)| \tag{2.6}$$

where $p^2(1-p)^n$ is the probability to obtain the subproduct AB^nA. Even if (2.6) were proved only in the Bernoulli case [13], the same argument extends to the Markovian case, leading to the following proposition.

Proposition 2.1. Let $\{\mathcal{P}_N\}$ be an infinite product of random matrices satisfying the Markovian distribution law (2.1) and (2.2), where A is a singular matrix given by (2.4) and B is general.

Then

$$\gamma = \frac{p_1}{p_1 + p_2} \log |\lambda| + \sum_{n=1}^{\infty} p_0 p_1 p_2 q_1{}^{n-1} \log |b_{11}(n)| \tag{2.7}$$

where $b_{11}(n)$ will be explicitly computed in section 4.

Remark 2.2. The proof of this proposition is analogous to that given in [13] once we notice that the probability to find the product AB^nA is $p_0 p_1 q_1^{n-1} p_2$ in the Markovian case whereas it is $p^2(1-p)^n$ in the Bernoulli case.

Remark 2.3. Notice that, in order to give an explicit value for γ, from (2.6) for the Bernoulli case or (2.7) for the Markovian case, the key point is the computation of $b_{11}(n)$, a problem which was not addressed in [13]. This is the object of the next sections

According to (2.7), it is possible to study the Bernoulli and the Markovian cases in the same manner, by writing

$$\gamma = \Pr(A) \log |\lambda| + L \sum_{n=1}^{\infty} x^{n-1} \log |b_{11}(n)| \tag{2.8}$$

where

$$\Pr(A) = p \qquad L = p^2(1-p) \qquad x = 1-p \tag{2.9}$$

in the Bernoulli case, and

$$\Pr(A) = \frac{p_1}{p_1 + p_2} \qquad L = \frac{p_1^2 p_2}{p_1 + p_2} \qquad x = q_1 \tag{2.10}$$

in the Markovian case.

3. The case of two singular matrices

In the case where B is singular, it is easy to calculate $b_{11}(n)$. Indeed, as explained above, we have two different cases, either

$$B = Q^{-1}\begin{pmatrix}\lambda_b & 0\\ 0 & 0\end{pmatrix}Q$$

or

$$B = Q^{-1}\begin{pmatrix}0 & \lambda_b\\ 0 & 0\end{pmatrix}Q$$

where Q is an invertible matrix and Q^{-1} is its inverse.

In the former case $\gamma = -\infty$, by the same argument as when A is given by (2.5).

In the latter case, if Q is written as $Q = \begin{pmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}$, then we have, for $n \geqslant 1$, $b_{11}(n) = {\lambda_b}^n q_{11}q_{22}$. And it is then easy to obtain γ; for a Bernoulli product, we have

$$\gamma = \gamma_{\mathrm{B}} = p\log|\lambda| + (1-p)\log|\lambda_b| + p(1-p)\log\left|\frac{b_{11}}{\mathrm{Tr}(B)}\right| \tag{3.1}$$

and for a Markovian product, we have

$$\gamma = \gamma_{\mathrm{M}} = \frac{p_1}{p_1+p_2}\log|\lambda| + \frac{p_2}{p_1+p_2}\log|\lambda_b| + \frac{p_1p_2}{p_1+p_2}\log\left|\frac{b_{11}}{\mathrm{Tr}(B)}\right| \tag{3.2}$$

where b_{11} denotes the first entry of the matrix B, and $\mathrm{Tr}(B)$ is the trace of B.

In the formulae (3.1) and (3.2), we notice a nonlinear term, originating in the non-commutativity of the matrices A and B.

4. Normal form and computation of B^n

When B is non-singular, we introduce a decomposition, which enables us to determine $b_{11}(n)$ for all positive integers n.

Let B be a real, non-singular 2×2 matrix, which we write as

$$B = |\det B|\mathcal{R}(-\varphi)\mathcal{B}\mathcal{R}(\varphi) \tag{4.1}$$

where $\det B$ is the determinant of B, $\mathcal{R}(\varphi) = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$ is a matrix rotation in the plane with an angle φ and $\mathcal{R}(-\varphi)$ is its inverse. The angle φ is determined by

$$\tan(2\varphi) = \frac{b_{22}-b_{11}}{b_{12}+b_{21}} \tag{4.2}$$

where b_{ij} are the entries of B.

We call the matrix $\mathcal{B}$, the normal form of B. We have four different types of normal forms $\mathcal{B}$, depending on the sign of the determinant and on the eigenvalues of B, which can be real, non-degenerated eigenvalues, real degenerated eigenvalues and conjugate complex eigenvalues.

We now define the quantity

$$\rho = \sqrt{\left|\frac{l_2}{l_1}\right|} \tag{4.3}$$

where

$$l_1 = b_{12}\sin^2(\varphi) - b_{21}\cos^2(\varphi) + \tfrac{1}{2}(b_{11}-b_{22})\sin(2\varphi)$$

and

$$l_2 = b_{12}\cos^2(\varphi) - b_{21}\sin^2(\varphi) - \tfrac{1}{2}(b_{11}-b_{22})\sin(2\varphi)$$

ρ will be used in the expression of $\mathcal{B}$.

Remark 4.1. Replacing B by $C = (1/\det B)B$, accroding to (2.3), γ is shifted by an additive constant, $\Pr(B)\log|\det B|$, where $\Pr(B)$ is the probability of the matrix B. Therefore, we can assume without loss of generality, that $|\det B| = 1$.

We now consider four different cases.

Case I: B is hyperbolic symplectic
The eigenvalues of B are real, non-degenerate and have the same sign, $\lambda_{1,2} = \epsilon \exp \pm\sigma$ with σ non-zero and $\epsilon = 1$ or -1; then, we get

$$\mathcal{B} = \epsilon \begin{pmatrix} \cosh\sigma & \rho\sinh\sigma \\ \dfrac{1}{\rho}\sinh\sigma & \cosh\sigma \end{pmatrix}. \tag{4.4}$$

Case II: B is hyperbolic, non-symplectic
In this case the eigenvalues of B are also real, non-degenerate but they have opposite signs, namely $\lambda_1 = \exp\sigma$ and $\lambda_2 = -\exp-\sigma$ or $-\lambda_1$ and $-\lambda_2$. Although we have two possible forms for $\mathcal{B}$, we can deduce one from the other by changing the sign of φ. We therefore retain only one form for $\mathcal{B}$:

$$\mathcal{B} = \begin{pmatrix} \sinh\sigma & \rho\cosh\sigma \\ \dfrac{1}{\rho}\cosh\sigma & \sinh\sigma \end{pmatrix}. \tag{4.5}$$

Case III: B is parabolic
The eigenvalues of B are given by $\lambda_{1,2} = \epsilon$, ($\epsilon = 1$ or -1) and we have either

$$\mathcal{B} = \epsilon \begin{pmatrix} 1 & \epsilon l_2 \\ 0 & 1 \end{pmatrix} \tag{4.6}$$

or

$$\mathcal{B} = \epsilon \begin{pmatrix} 1 & 0 \\ -\epsilon l_1 & 1 \end{pmatrix}. \tag{4.7}$$

Notice that we can pass from (4.7) to (4.6) by a rotation in the plane with an angle $\pi/2$. In the following, we define the normal form of a parabolic matrix as given by (4.6).

Case IV: B is elliptic
The eigenvalues of B are complex conjugate, $\lambda_{1,2} = \exp\pm i\sigma$, in which case

$$\mathcal{B} = \begin{pmatrix} \cos\sigma & -\rho\sin\sigma \\ \frac{1}{\rho}\sin\sigma & \cos\sigma \end{pmatrix}. \tag{4.8}$$

Therefore, if B is a non-singular matrix, we can write it in the form given by (4.1) where $\mathcal{B}$ is given by one of the expressions (4.4), (4.5), (4.6) or (4.8).

We can now easily compute $\mathcal{B}^n$ in each of the previous cases. We summarize the result in the following proposition.

Proposition 4.2. Let B be a non-singular matrix with a normal form $\mathcal{B}$ defined by (4.1). Then

$$B^n = \mathcal{R}(-\varphi)\mathcal{B}^n\mathcal{R}(\varphi)\,. \tag{4.9}$$

If $\mathcal{B}$ is symplectic hyperbolic, given by (4.4), then

$$\mathcal{B}^n = \epsilon^n \begin{pmatrix} \cosh n\sigma & \rho\sinh n\sigma \\ \frac{1}{\rho}\sinh n\sigma & \cosh n\sigma \end{pmatrix}. \tag{4.10}$$

If B is non-symplectic hyperbolic, given by (4.5), then either

$$\mathcal{B}^n = \begin{pmatrix} \cosh n\sigma & \rho\sinh n\sigma \\ \frac{1}{\rho}\sinh n\sigma & \cosh n\sigma \end{pmatrix} \tag{4.11}$$

if n is odd; or

$$\mathcal{B}^n = \begin{pmatrix} \sinh n\sigma & \rho\cosh n\sigma \\ \frac{1}{\rho}\cosh n\sigma & \sinh n\sigma \end{pmatrix} \tag{4.12}$$

if n is even.

If B is parabolic, given by (4.6), then

$$\mathcal{B}^n = \epsilon^n \begin{pmatrix} 1 & n\epsilon^n l_2 \\ 0 & 1 \end{pmatrix}. \tag{4.13}$$

If B is elliptic, given by (4.8), then

$$\mathcal{B}^n = \begin{pmatrix} \cos n\sigma & -\rho\sin n\sigma \\ \frac{1}{\rho}\sin n\sigma & \cos n\sigma \end{pmatrix}. \tag{4.14}$$

5. Lyapunov exponent: explicit formulae

We are now ready to perform the analysis of the formula (2.8) in the case where the matrix B is non-singular.

As previously, without loss of generality, we suppose that $|\det B| = 1$.

We now give the value of the largest Lyapunov exponent in each of the four cases treated in the previous section.

Case I: B is a hyperbolic symplectic matrix
For $n \geqslant 1$, we have

$$b_{11}(n) = \epsilon^n[\cosh(n\sigma) + \cosh(\alpha)\sinh(n\sigma)\sin(2\varphi)] \tag{5.1}$$

where

$$\cosh\alpha = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right). \tag{5.2}$$

In (5.1), we can take $\sigma > 0$. Indeed, if σ is negative, we change the sign of φ and use $-\sigma$ instead of σ.

Notice first that if there exists an integer $n_0 \geqslant 1$ such that $b_{11}(n_0) = 0$ then $\gamma = -\infty$. The condition $b_{11}(n_0) = 0$ is equivalent to

$$\cosh\alpha = -\frac{1}{\sin(2\varphi)\tanh(n_0\sigma)}$$

and therefore we can construct some products for which $\gamma = -\infty$.

On the contrary, if we suppose that for all integers $n \geqslant 1$, $b_{11}(n) \neq 0$, and we define the quantities

$$\delta = 1 + \sin(2\varphi)\cosh(\alpha) \tag{5.3}$$

and

$$\tau = \frac{1 - \sin(2\varphi)\cosh(\alpha)}{1 + \sin(2\varphi)\cosh(\alpha)}. \tag{5.4}$$

We will use τ and δ in the expression of γ.

We now distinguish two cases.

(a) $\delta = 0$. A straightforward calculation gives

$$\gamma = \gamma_{\mathrm{B}} = p\log|\lambda| - (1-p)\sigma \tag{5.5}$$

for the Lyapunov exponent in the Bernoulli case, and

$$\gamma = \gamma_{\mathrm{M}} = \frac{p_1}{p_1 + p_2}\log|\lambda| - \frac{p_2}{p_1 + p_2}\sigma \tag{5.6}$$

for the Markovian case.

(b) $\delta \neq 0$. Depending on whether τ is zero or not, we have different expressions for γ.
(i) $\tau = 0$: we easily obtain the largest Lyapunov exponent for a Bernoulli product:

$$\gamma = \gamma_{\mathrm{B}} = p\log|\lambda| + (1-p)\sigma \tag{5.7}$$

and for a Markovian product:

$$\gamma = \gamma_{\mathrm{M}} = \frac{p_1}{p_1 + p_2}\log|\lambda| + \frac{p_2}{p_1 + p_2}\sigma \tag{5.8}$$

(ii) $\tau \neq 0$: Here we obtain γ as a convergent series given by

$$\gamma = \Pr(A)\log|\lambda| + \frac{L}{(1-x)^2}\sigma + \frac{L}{1-x}\log\frac{|\delta|}{2} + L\sum_{n=1}^{\infty} x^{n-1}\log|1 + \tau\,\mathrm{e}^{-2n\sigma}| \tag{5.9}$$

where L is defined by (2.9) and (2.10).

Notice that if γ_N is the sum of the Nth first terms in (5.9) of γ, the error is

$$|\gamma - \gamma_N| \leqslant |\tau|\exp(-2(N+1)\sigma)Lx^{N+1} \tag{5.10}$$

and therefore it is exponentially small since $0 < x < 1$.

Case II: B is a hyperbolic no-symplectic matrix
This case is similar to the previous one with only a slight difference concerning (5.9). By applying the decomposition given by (4.1) we obtain

$$b_{11}(n) = \frac{1+\sin(2\varphi)\cosh(\alpha)}{2}\, e^{n\sigma} + (-1)^n \frac{1-\sin(2\varphi)\cosh(\alpha)}{2}\, e^{-n\sigma} . \tag{5.11}$$

Again, we can suppose that $\sigma > 0$ without loss of generality. And, as above, if there exists an $n_0 \geqslant 1$ such that $b_{11}(2n_0) = 0$ or $b_{11}(2n_0 - 1) = 0$, then $\gamma = -\infty$.

Recall that δ and τ are defined by (5.3) and (5.4).

If $\delta = 0$ or if $\delta \neq 0$ but $\tau = 0$, the largest Lyapunov exponent is given by the same expressions as in the previous case: (5.5) and (5.6), or (5.7) and (5.8).

Instead, if $\delta \neq 0$ and $\tau \neq 0$, we obtain

$$\gamma = \Pr(A)\log|\lambda| + \frac{L}{(1-x)^2}\sigma + \frac{L}{1-x}\log\left|\frac{\delta}{2}\right| + L\sum_{n=1}^{\infty} x^{n-1}\log|1+(-1)^n\tau\, e^{-2n\sigma}| . \tag{5.12}$$

This series is convergent, indeed we have

$$|\gamma - \gamma_N| \leqslant L x^{N+1}\Delta \tag{5.13}$$

where

$$\Delta = \max(|\log|1+(-1)^{N+1} e^{-2(N+1)\sigma}||, |\log|1+(-1)^{N+2} e^{-2(N+2)\sigma}||) . \tag{5.14}$$

Case III: B is a parabolic matrix
The normal form of B is

$$\mathcal{B} = \epsilon \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

and thus, for $n \geqslant 1$,

$$b_{11}(n) = \epsilon^n\left[1 + \tfrac{1}{2}nb\sin(2\varphi)\right] . \tag{5.15}$$

We suppose that $b_{11} \neq 0$ for all integers $n \geqslant 1$. The trivial case where $\sin(2\varphi) = 0$ corresponds to an infinite product of diagonal and triangular matrices. In this case,

$$\gamma = \Pr(A)\log|\lambda| .$$

When, instead, $\sin(2\varphi) \neq 0$, we have, for a Bernoulli product,

$$\gamma = p\log|\lambda| + p(1-p)\log\left|\frac{b\sin(2\varphi)}{2}\right| + p^2(1-p)\sum_{n=1}^{\infty}(1-p)^{n-1}\log\left|n + \frac{1}{nb\sin(2\varphi)}\right| \tag{5.16}$$

and, for a Markovian product,

$$\gamma = \frac{p_1}{p_1 + p_2} \log|\lambda| + \frac{p_1 p_2}{p_1 + p_2} \log\left|\frac{b\sin(2\varphi)}{2}\right| + \frac{p_1^2 p_2}{p_1 + p_2} \sum_{n=1}^{\infty} (1 - p_1)^{n-1} \log\left|n + \frac{1}{b\sin(2\varphi)}\right|. \tag{5.17}$$

These series are convergent since we have

$$|\gamma - \gamma_N| \leqslant |\log\left|1 + \frac{2}{nb\sin(\varphi)}\right|\left|\frac{1}{1-x} + \frac{N+1}{(1-x)^2}\right| x^N \tag{5.18}$$

and $0 < x < 1$.

Case IV: B is an elliptic matrix

In this case we also obtain series for γ, but it is very difficult, in general, to evaluate the rest of the corresponding partial sums. Indeed, in this situation, as expected, the summability of the series in the expression of γ is related to the arithmetic properties of σ, when the latter is irrational (mod 2π). On the another hand, if σ is rational (mod 2π), γ may be given as a sum of a finite number of terms.

As above, by using the decomposition (4.1), we obtain

$$b_{11}(n) = \cos(n\sigma) + \sinh(\alpha)\sin(n\sigma)\sin(2\varphi) \tag{5.19}$$

where

$$\sinh(\alpha) = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right) \tag{5.20}$$

with $\alpha \in \mathbb{R}$.

If $b_{11}(n_0) = 0$ for some positive integer n_0, then $\gamma = -\infty$.

Suppose now that $b_{11}(n) \neq 0$, for all positive integers n, and σ is rational (mod 2π), i.e. $\sigma = (r/s)2\pi$ where $r, s \in \mathbb{N}^*$, ($r < s$) are irreducible. Thus

$$\gamma = p\log|\lambda| + \frac{p^2}{1 - (1-p)^s} \sum_{j=1}^{s-1} (1-p)^j \log|b_{11}(j)| \tag{5.21}$$

for a Bernoulli product, and

$$\gamma = \frac{p_1}{p_1 + p_2} \log|\lambda| + \frac{p_1^2 p_2}{(p_1 + p_2)(1 - (1 - p_1)^s)} \sum_{j=1}^{s-1} (1 - p_1)^{j-1} \log|b_{11}(j)| \tag{5.22}$$

where

$$b_{11}(j) = \cos\left(\frac{2jr}{s}\pi\right) + \sinh\alpha\sin(2\varphi)\sin\left(\frac{2jr}{s}\pi\right) \tag{5.23}$$

for a Markovian product.

6. One-parameter family of products: an example

One may wonder if a limit procedure may permit the computation of γ in the elliptic case for irrational σ from the corresponding formulae (5.21) or (5.23), when σ is rational.

More generally, the continuity of the largest Lyapunov exponent as a function of the amplitude of the disorder in disordered systems is known to be an important issue. In this section, we illustrate the results of the previous sections by giving an example where, indeed, γ is a discontinuous function of a parameter α.

We consider the family $\{P_\infty(\alpha) = (A\ ,B(\alpha)\ ,\ p_1\ ,\ p_2),\ \alpha \in \mathbb{R}^{+*},\ p_1 > 0\ ,\ p_2 > 0\}$ of infinite Markovian products of the singular matrix $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and a family $B(\alpha)$ of matrices, depending on a parameter α. p_1 and p_2 are the transition probabilities defined by (2.1).

If $p_1 + p_2 = 1$, then we recover the case of a Bernoulli product.

$B(\alpha)$ is defined by

$$B(\alpha) = \mathcal{R}(-\varphi)\begin{pmatrix} \cosh\sigma_0 & \rho\sinh\sigma_0 \\ \frac{1}{\rho}\sinh\sigma_0 & \cosh\sigma_0 \end{pmatrix}\mathcal{R}(\varphi) \tag{6.1}$$

where $\varphi = -\pi/4$, $\sigma > 0$ is a fixed parameter; and $\rho = \mathrm{e}^\alpha$ with $\alpha \in \mathbb{R}^{+*}$.

The condition $b_{11}(\alpha) = 0$ gives us

$$\alpha = \alpha_n = -\mathrm{Argch}(\coth(n\sigma)) \tag{6.2}$$

and $\gamma(\alpha_n) = -\infty$.

Now for all $\alpha \in \mathbb{R}^{+*}$, we define

$$\begin{aligned} f(\alpha) = \gamma(\alpha) = {} & \frac{p_2}{p_1+p_2}\sigma_0 + \frac{p_1 p_2}{p_1+p_2}\log\left(\frac{\cosh\alpha - 1}{2}\right) \\ & + \frac{p_1^2 p_2}{p_1+p_2}\sum_{n=1}^{\infty}(1-p_1)^{n-1}\log\left(1 + \frac{\cosh\alpha+1}{\cosh\alpha-1}\,\mathrm{e}^{-2n\sigma_0)}\right). \end{aligned} \tag{6.3}$$

Since the series in (5.3) is summable, then $f(\alpha)$ is defined and continuous.

But since, for $\alpha \neq \alpha_n$, we have

$$\gamma(\alpha) = f(\alpha)\,. \tag{6.4}$$

We obtain

$$\lim_{\alpha\to\alpha_n}\gamma(\alpha) = f(\alpha_n) \neq \gamma(\alpha_n) \tag{6.5}$$

for all $n \in \mathbb{N}^*$. Thus $\gamma(\alpha)$ is a discontinuous function of α, for each $\alpha = \alpha_n$, $n \in \mathbb{N}$.

7. Concluding remarks

To conclude, we have been able to compute explicit formulae for the Lyapunov exponents of some infinite products of random matrices, a task previously performed only in a few cases [10, 11].

Since we deal with products constructed with two different matrices, it was natural to generalize Pincus formula [equation (2.6) of 13], to the more general case of Markovian distribution. This is the content of (2.7). It is seen from these two formulae, (2.6) and (2.7), that in order to achieve the computation of the Lyapunov exponent, the knowledge of the first entry of all integer powers of the matrice B is sufficient. This is done using a special form of B, defined in (4.1). We then compute the B^n, $n > 1$, and therefore the corresponding entries $b_{11}(n)$, in different cases: B being a hyperbolic (symplectic or not), a parabolic or an elliptic matrix. For each of these cases, the formulae (2.6) and (2.7) allow us to get an explicit series for the Lyapunov exponent.

For the hyperbolic, as well as parabolic case, we get exponentially fast convergent series. Instead, the corresponding sum for the elliptic matrix leads to a series for which the rate of convergence is related to arithmetic properties of the eigenvalues of B. The case of rational eigenvalues give rise, of course, to a finite sum.

Finally, we show how, in the last case (elliptic matrix), there is no continuity arguments allowing us to approach the Lyapunov exponent for irrational eigenvalues by the corresponding exponents of rational approximations.

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